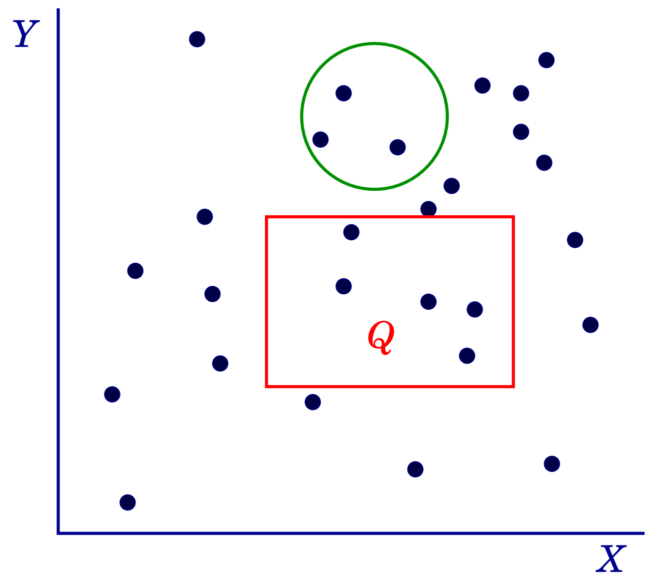
Range Searching

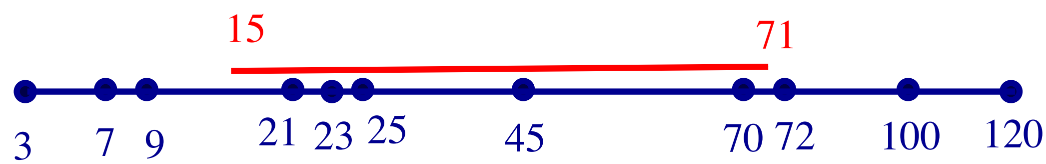
- Data structure for a set of objects (points, rectangles, polygons) for efficient range queries.



- Depends on type of objects and queries. Consider basic data structures with broad applicability.
- Time-Space tradeoff: the more we preprocess and store, the faster we can solve a query.
- Consider data structures with (nearly) linear space.

1-Dimensional Search

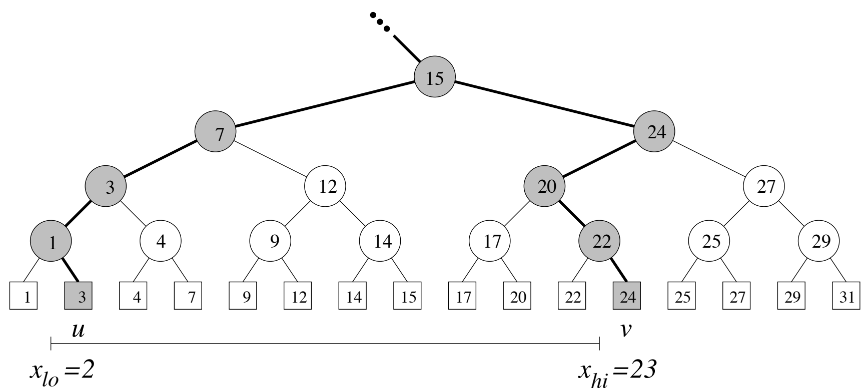
- **Points in 1D** $P = \{p_1, p_2, \ldots, p_n\}$**.**
- **Queries are intervals.**



- **If the range contains** k **points, we want to solve the problem in** $O(\log n + k)$ **time.**

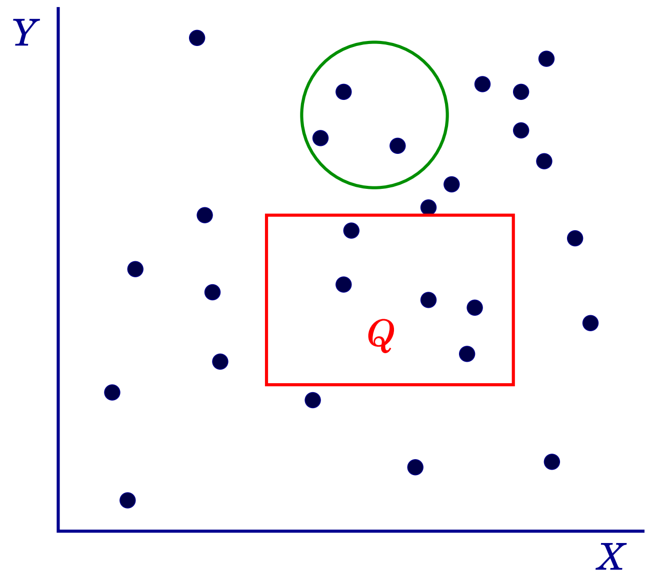
- **Does hashing work? Why not?**
- **A sorted array achieves this bound. But it doesn't extend to higher dimensions.**
- **Instead, we use a balanced binary tree.**

Tree Search



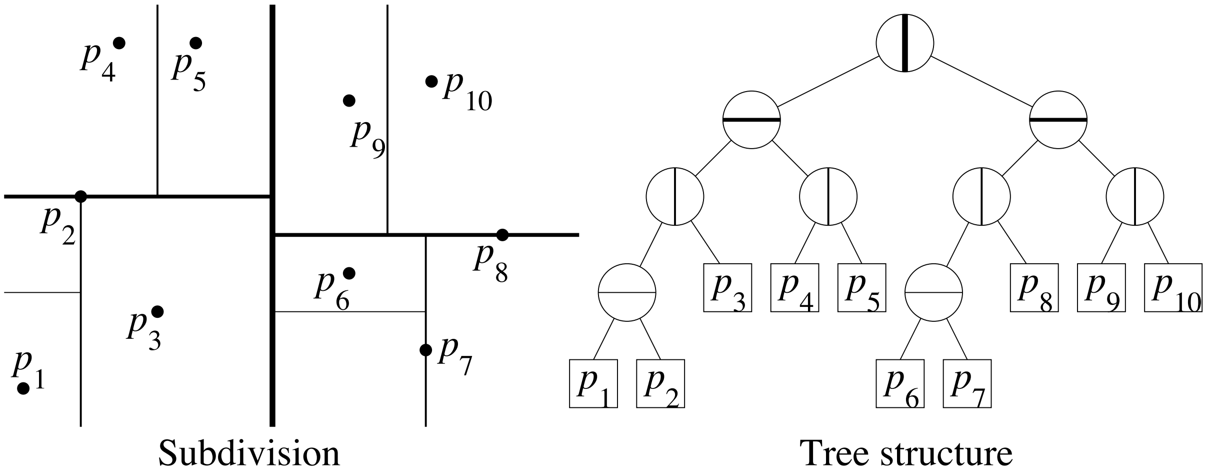
- **Build a balanced binary tree on the sorted list of points (keys).**
- **Leaves correspond to points; internal nodes are branching nodes.**
- **Given an interval $[x_{lo}, x_{hi}]$, search down the tree for x_{lo} and x_{hi}.**
- **All leaves between the two form the answer.**
- **Tree searches takes $2 \log n$, and reporting the points in the answer set takes $O(k)$ time; assume leaves are linked together.**

Multi-Dimensional Data



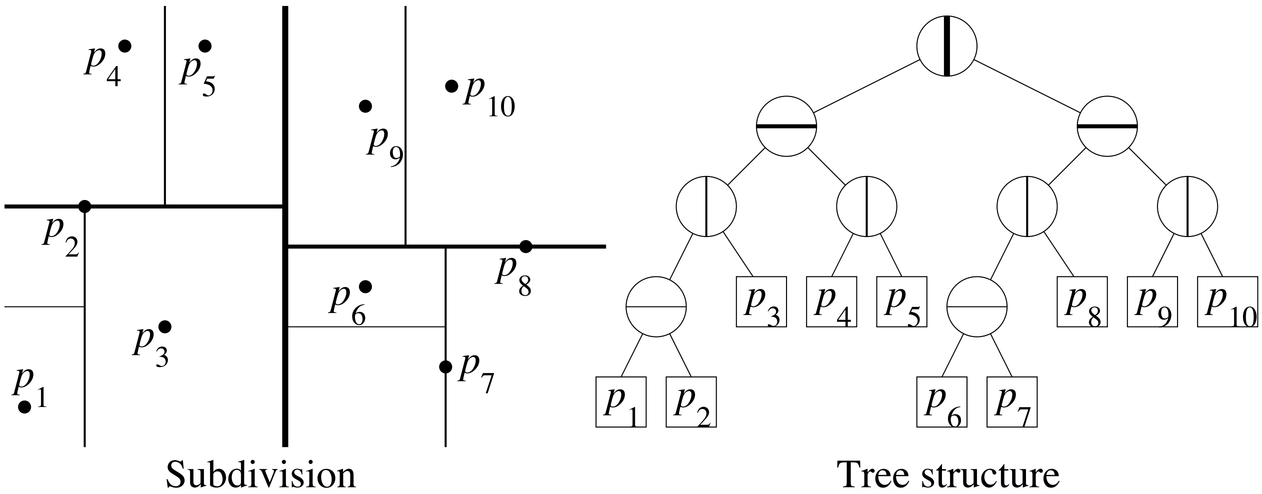
- **Range searching in higher dimensions?**
- kD**-trees [Jon Bentley 1975]. Stands for** k**-dimensional trees.**
- **Simple, general, and arbitrary dimensional. Asymptotic search complexity not very good.**
- **Extends 1D tree, but alternates using** x**-** y**-coordinates to split. In** k**-dimensions, cycle through the dimensions.**

kD-Trees



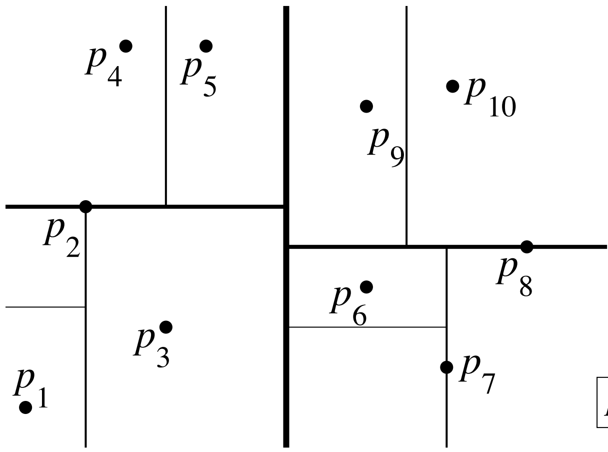
- **A binary tree. Each node has two values: split dimension, and split value.**
- **If split along x, at coordinate s, then left child has points with x-coordinate $\leq s$; right child has remaining points. Same for y.**
- **When $O(1)$ points remain, put them in a leaf node.**
- **Data points at leaves only; internal nodes for branching and splitting.**

Splitting

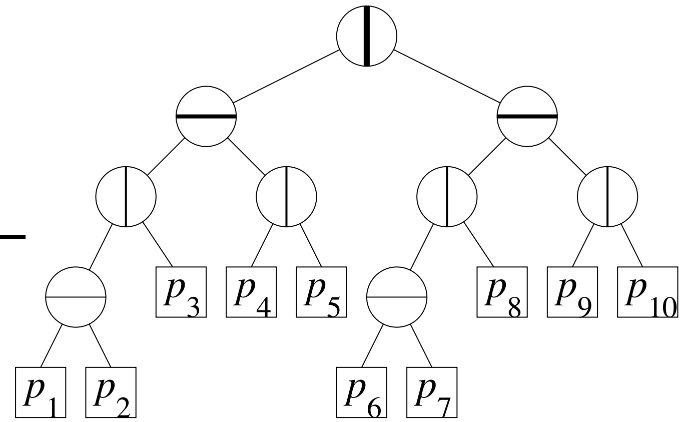


- To get balanced trees, use the median coordinate for splitting—median itself can be put in either half.
- With median splitting, the height of the tree guaranteed to be $O(\log n)$.
- Either cycle through the splitting dimensions, or make data-dependent choices. E.g. select dimension with max spread.

Space Partitioning View



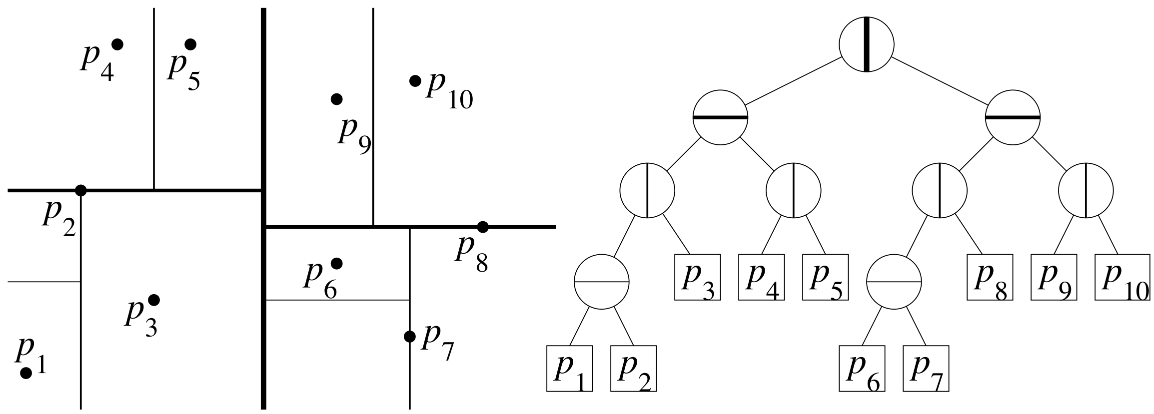
Subdivision



Tree structure

- kD-tree induces a space subdivision—each node introduces a x- or y-aligned cut.
- Points lying on two sides of the cut are passed to two children nodes.
- The subdivision consists of rectangular regions, called cells (possibly unbounded).
- Root corresponds to entire space; each child inherits one of the halfspaces, so on.
- Leaves correspond to the terminal cells.
- Special case of a general partition BSP.

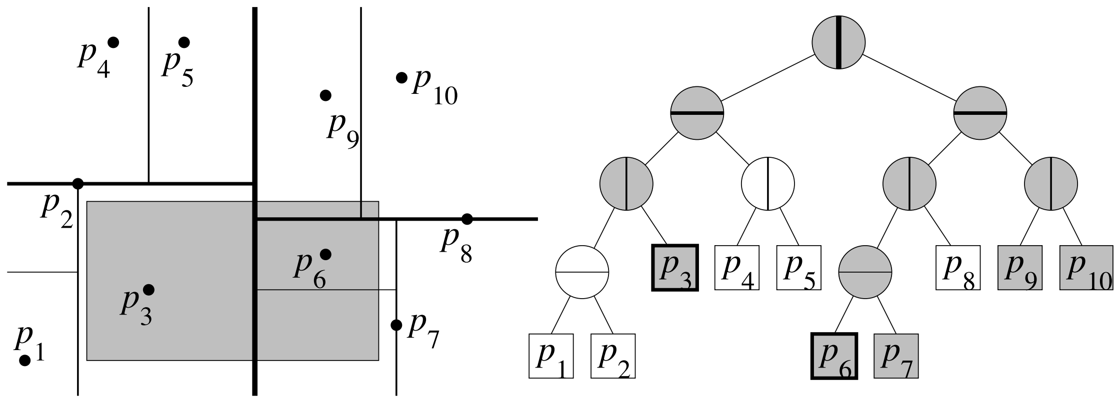
Construction



Subdivision　　Tree structure

- **Can be built in $O(n \log n)$ time recursively.**
- **Presort points by x and y-coordinates, and cross-link these two sorted lists.**
- **Find the x-median, say, by scanning the x list. Split the list into two. Use the cross-links to split the y-list in $O(n)$ time.**
- **Now two subproblems, each of size $n/2$, and with their own sorted lists. Recurse.**
- **Recurrence $T(n) = 2T(n/2) + n$, which solves to $T(n) = O(n \log n)$.**

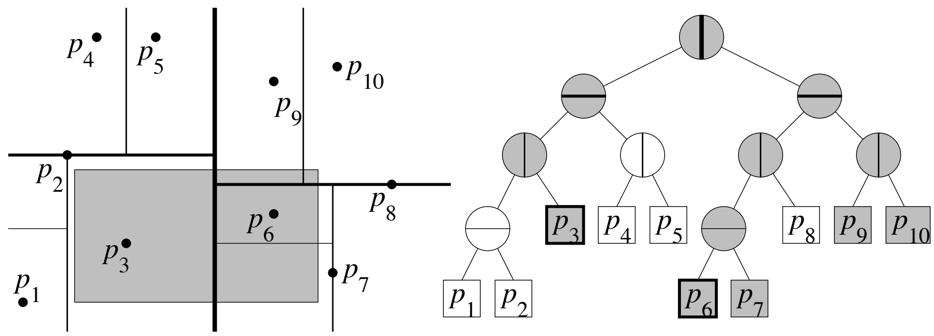
Searching kD-Trees



The range | Nodes visited in search

- **Suppose query rectangle is R. Start at root node.**
- **Suppose current splitting line is vertical (analogous for horizontal). Let v, w be left and right children nodes.**
- **If v a leaf, report $cell(v) \cap R$;**
  if $cell(v) \subseteq R$, report all points of cell(v);
  if $cell(v) \cap R = \emptyset$, skip;
  otherwise, search subtree of v recursively.
- **Do the same for w.**
- **Procedure obviously correct. What is the time complexity?**

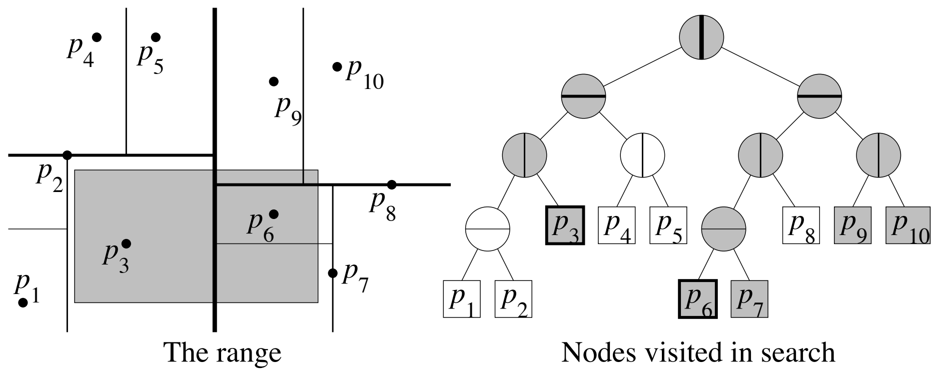
Search Complexity



The range   Nodes visited in search

- **When $cell(v) \subseteq R$, complexity is linear in output size.**
- **It suffices to bound the number of nodes v visited for which the boundaries of $cell(v)$ and R intersect.**
- **If $cell(v)$ outside R, we don't search it; if $cell(v)$ inside R, we enumerate all points in region of v; a recursive call is made only if $cell(v)$ partially overlaps R; the kD-tree height is $O(\log n)$.**
- **Let ℓ be the line defining one side of R.**
- **We prove a bound on the number of cells that intersect ℓ; this is more than what is needed; multiply by 4 for total bound.**

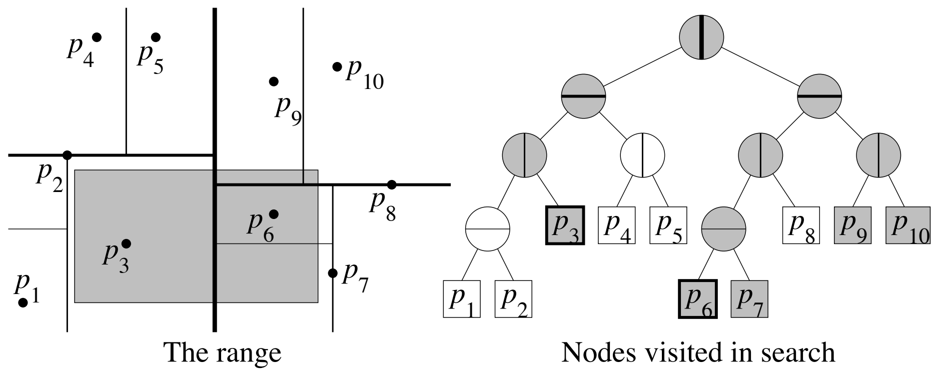
Search Complexity



- **How many cells can a line intersect?**
- **Since splitting dimensions alternate, the key idea is to consider two levels of the tree at a time.**
- **Suppose the first cut is vertical, and second horizontal. We have 4 cells, each with $n/4$ points.**
- **A line intersects exactly two cells; the others cells will be either outside or entirely inside R.**
- **The recurrence is**

$$Q(n) = \begin{cases} 1 & \textbf{if } n = 1, \\ 2Q(n/4) + 2 & \textbf{otherwise.} \end{cases}$$

Search Complexity



- **The recurrence $Q(n) = 2Q(n/4) + 2$ solves to**
$$Q(n) = O(\sqrt{n})$$

- **kD-Tree is an $O(n)$ space data structure that solves 2D range query in worst-case time $O(\sqrt{n} + m)$, where m is the output size.**

d-Dim Search Complexity

- **What's the complexity in higher dimensions?**
- **Try 3D, and then generalize.**
- **The recurrence is**

$$Q(n) = 2^{d-1} Q(n/2^d) + 1$$

- **It solves to**

$$Q(n) = O(n^{1-1/d})$$

- kD**-Tree is an** $O(dn)$ **space data structure that solves** d**-dim range query in worst-case time** $O(n^{1-1/d} + m)$**, where** m **is the output size.**