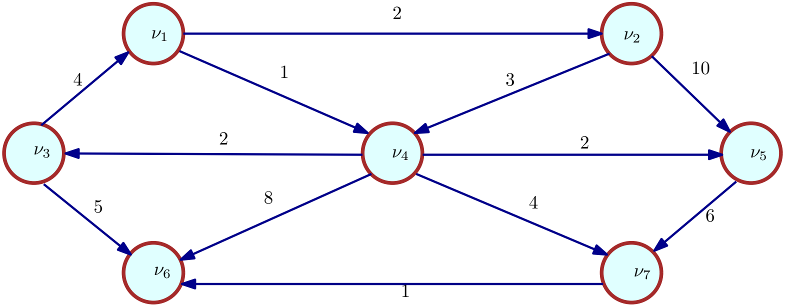
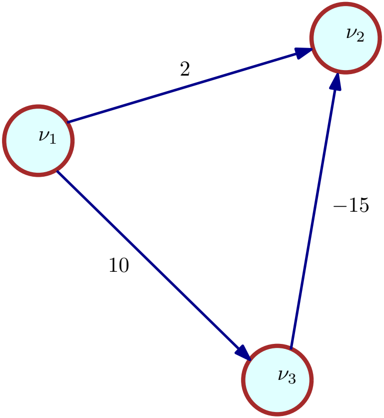
The Shortest Path problem

- Given graph and a vertex s find shortest paths from s to all other vertices.
- Map routing, robot navigation, urban traffic planning
- Optimal pipelining of VLSI chip
- Routing of telecommunication messages
- Network routing protocols (OSPF, BGP, RIP)
- Seam carving, texture mapping, typesetting in TeX!

Example with positive edge weights



Example with negative edge weights



Unweighted shortest paths

- Given unweighted graph G
- Can assume all edge weights are 1
- Find shortest paths from s
- There is what is known as a shortest path tree!
- Can be found using Breadth First Search (BFS)

Naive implementation: pseudo code

```
void Graph::unweighted( Vertex s ){
   Vertex v,w;
   s.dist = 0;
   for(int currDist=0; currDist < NUM_VERTICES; currDist++)
     for each vertex v
       if( !v.known && v.dist == currDist ){
         v.known = true;
         for each w adjacent to v
         if( w.dist == INFINITY ){
           w.dist = currDist + 1;
           w.path = v;
         }
       }
}
```

Smarter implementation: pseudo code

```
void Graph::unweighted( Vertex s ){
  Queue q( NUM_VERTICES );
  Vertex v,w;
  q.enqueue(s);
  s.dist = 0;
  while( !q.isEmpty() ){
    v = q.dequeue();
    v.known = true;
    for each w adjacent to v
      if( w.dist == INFINITY ){
        w.dist = v.dist + 1;
        w.path = v;
        q.enqueue( w );
      }
   }
}
```

Main structural properties of Shortest Paths

- Prefixes of shortest paths are themselves shortest paths
- Does a shortest path always exist?
- What about a shortest path tree?
- How can we compute such a tree

Main concepts

- *known* vertices
- Relaxation of an edge $(v, w) : d(w) = \min(d(w), d(v) + c_{vw})$
- Next: The Djikstra algorithm

Edsger W. Dijkstra: select quotes

" Do only what only you can do. "

" In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind. "

" The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence. "

" It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration. "

" APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums. "

Edsger W. Dijkstra
Turing award 1972

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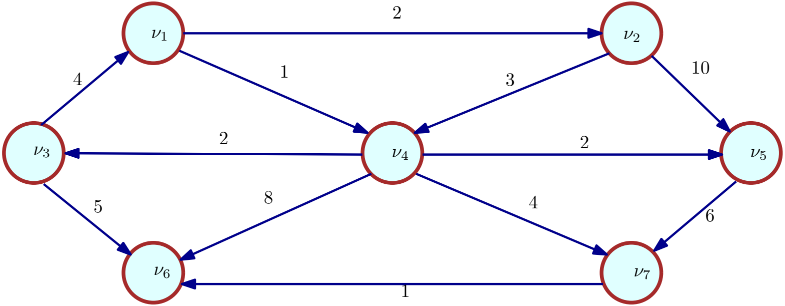
Edsger W. Dijkstra: select quotes

Djikstra algorithm : arbitrary non-negative edge weights

- Store $d_v, known, p_v$
- Pick vertex with minimum d_v (that is not *known*)
- Relax all edges outgoing from it
- Repeat until all vertices are known

Example of Djikstra in action



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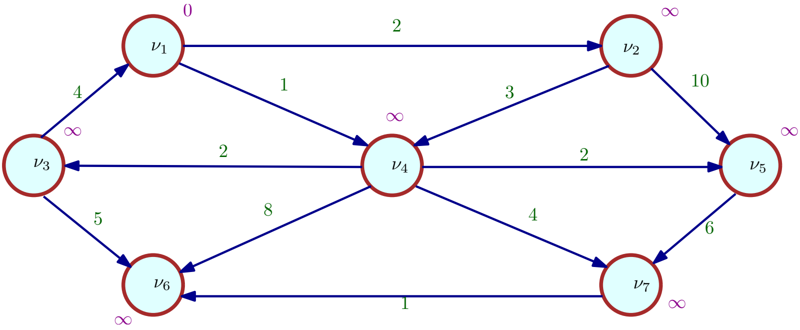

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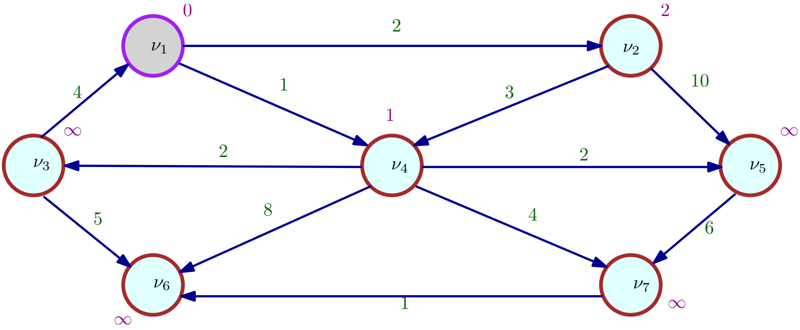

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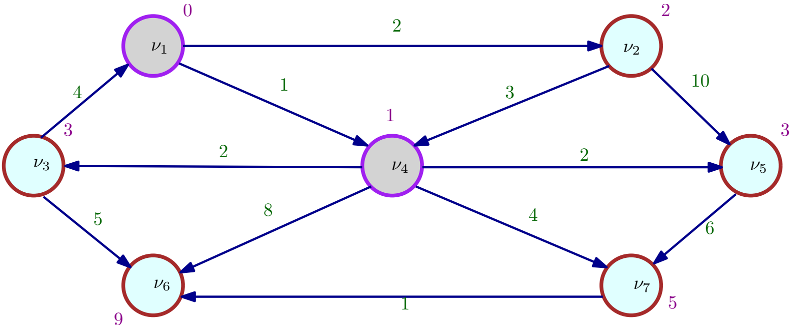

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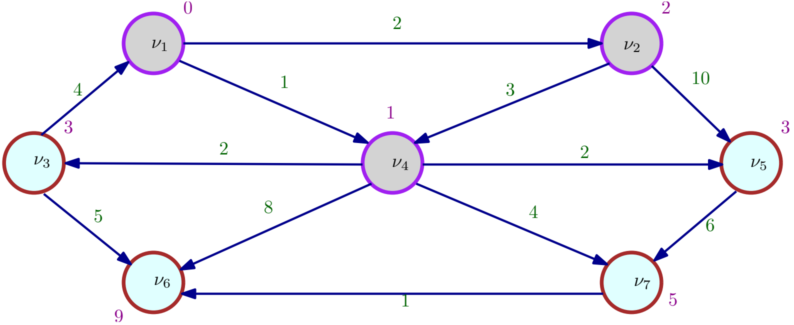

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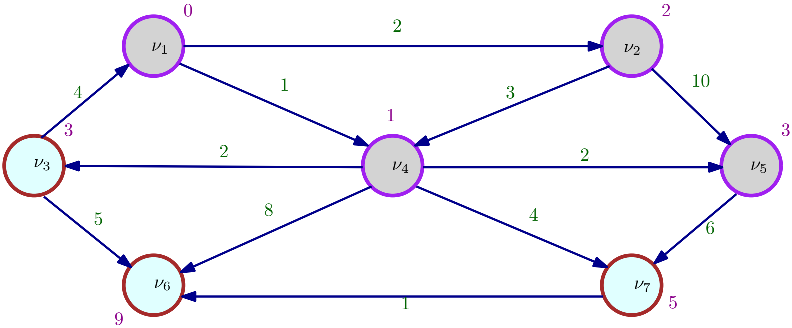

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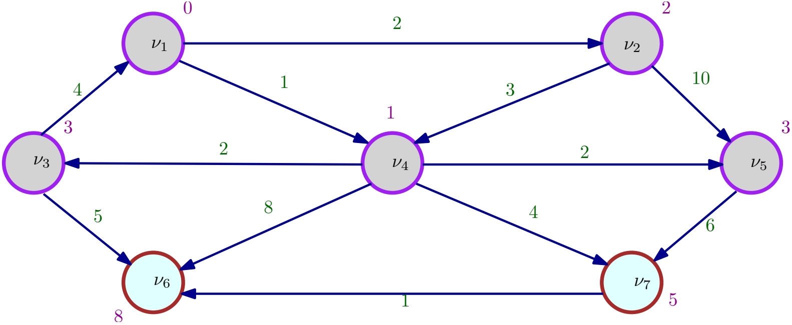

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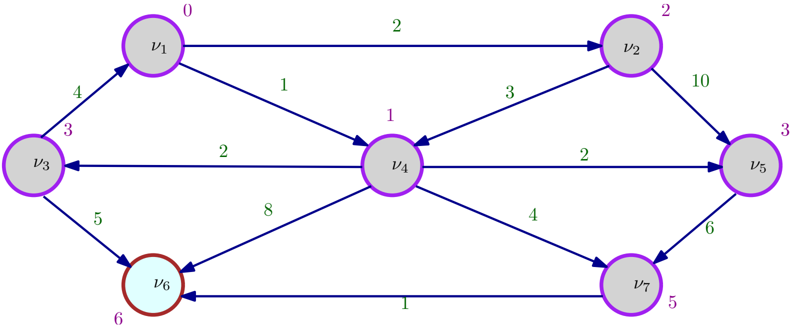

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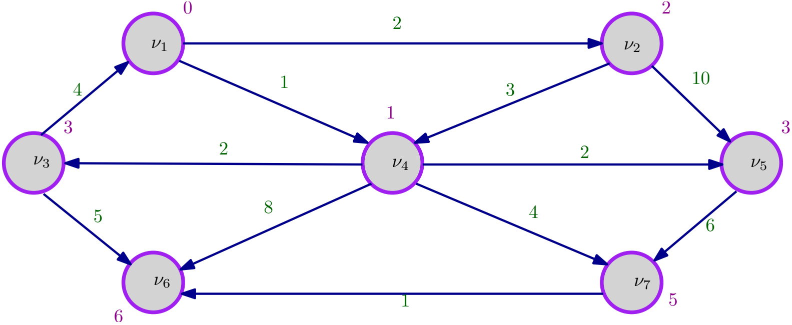

Djikstra data-structures

```
struct Vertex
{
  List adj;   // Adjacency list
  bool known;
  DistType dist;
  Vertex path;  // ref to parent in path
};

void Graph::createTable( vector<Vertex> & t){
  readGraph( t ); //Read graph, fill in adj
  for(int i=0; i < t.size(); i++){
    t[i].known = false;
    t[i].dist = INFINITY;
    t[i].path = NOT_A_VERTEX;
  }
  NUM_VERTICES = t.size();
}
```

Shortest Paths after Djikstra run

```
void Graph::printPath( Vertex v )
{
  if(v.path != NOT_A_VERTEX)
  {
    printPath( v.path );
    cout << " to ";
  }
  cout << v;
}
```

The Djikstra algorithm: pseudo-code

```
void Graph::djikstra( Vertex s ){
  Vertex v,w;
  s.dist = 0;
  for( ; ; ){
    v = smallest unknown distance vertex;
    if( v == NOT_A_VERTEX )
      break;
    v.known = true;
    for each w adjacent to v;
    if( !w.known )
      if( v.dist + c(v,w) < w.dist ){
        decrease w.dist to v.dist + c(v,w);
        w.path = v;
      }
  }
}
```

Implementing Djikstra

- Naive implementation (using array to find min d_v) : $O(|E| + |V^2|) = O(|V|^2)$
- Could we be better for sparse graphs?

Implementing Djikstra

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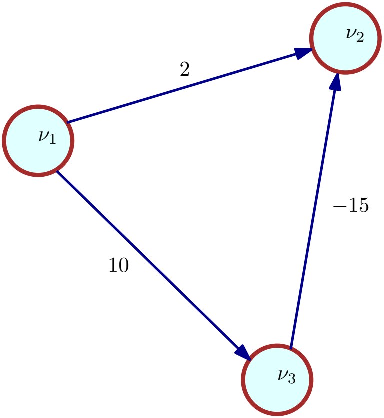
- Could we be better for sparse graphs?

PQ impl	insert	delete-min	decrease-key	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$\log_{\frac{E}{V}} V$
Fibonacci heap	1	$\log V$	1	$E + V \log V$

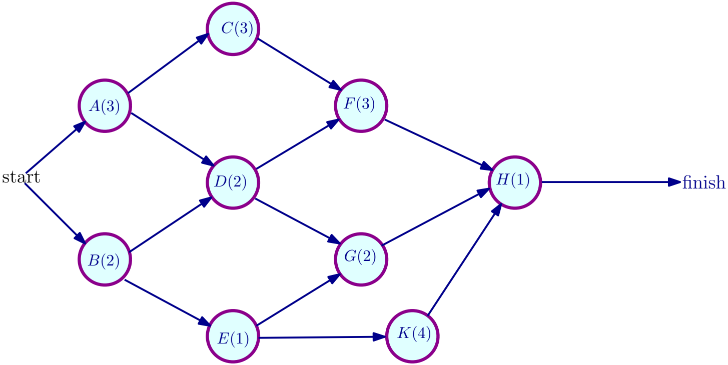
Negative edge weights!



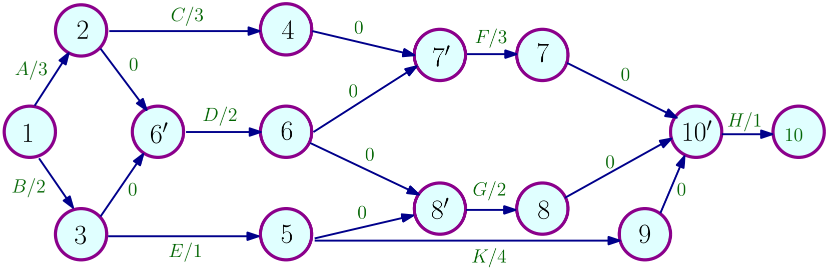
Acyclic Graphs

- Important special case : Nonreversible chemcial reactions, critical path analysis
- Running time is $O(|E| + |V|)$
- Djikstra can be implemented along with Topological sort

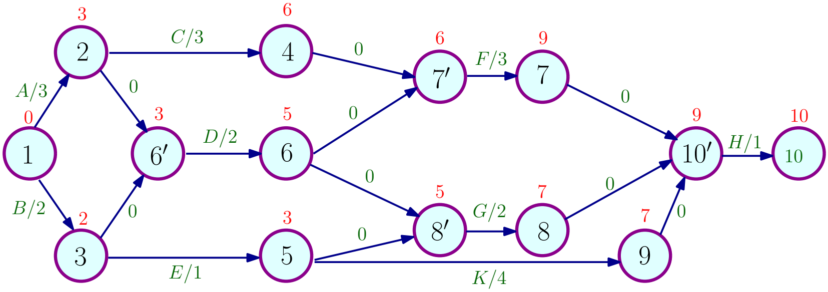
Example: Activity-node graph



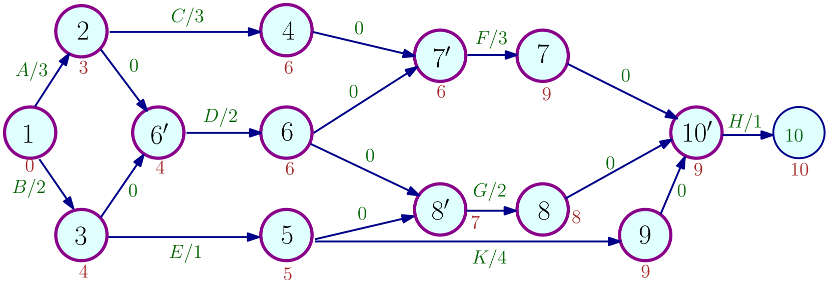
Event-node graph



Earliest Completion times



Latest Completion times



EC, LC, Slack, Critical Path

Earliest and Latest Completion times

$$EC_1 = 0$$
$$EC_w = \max_{(v,w) \in E} (EC_v + c_{v,w})$$

$$LC_n = EC_n$$
$$LC_v = \min_{(v,w) \in E} (LC_w - c_{v,w})$$

Slack of an edge (v, w)

$Slack_{(v,w)} = LC_w - EC_v - c_{(v,w)}$

The Bellman Ford algorithm

Basic Pseudo code

$d[s] = 0$
for $i = 1$ to $|V|$
    Relax each edge

- Why does this work?

Bellman Ford: Queue Based Implementation

```
void Graph::weightedNegative( Vertex s ){
  Queue q(NUM_VERTICES);
  Vertex v,w;
  q.enqueue(s);
  s.dist = 0;
  while(! q.isEmpty()){
    v=q.dequeue();
    for each w adjacent to v
      if(v.dist + c(v,w) < w.dist){
        w.dist = v.dist + c(v,w);
        w.path = v;
        if(w is not already in q)
          q.enqueue(w);
      }
  }
}
```

Bellman Ford contd.

- Runtime is $O(EV)$
- Can be used to detect negative cycles
- Useful in finding arbitrage opportunities!

All-Pairs Shortest Path

- Can run $|V|$ Djikstra's - $O(|E||V|\log|V|)$
- Floyd Warshall : Dynamic programming algorithm
- Works in $O(|V|^3)$